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# Non-abelian self-dual strings in six dimensions from four dimensional 1/2-BPS monopoles

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## Abstract

We explain a new construction of self-dual string solutions to the non-abelian two-form self-duality equation proposed in [1]. This class of self-dual strings is determined by the BPS monopoles in four-dimensions and the self-dual string charge is given by the charge of the monopole. Our construction covers the  $SO(4)$  invariant self-dual string solutions found previously. We have also constructed, based on the 't Hooft–Polyakov monopole, a singular solution that describes two finitely separated M5-branes meeting midway in between. We comment that as BPS monopoles are generally given by the Nahm construction, our construction suggests that a generalized Nahm transform may exist for the non-abelian self-dual strings.

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**Keywords:** M-Theory; D-branes; M-branes; Gauge symmetry

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## 1. Introduction

The theory of  $N$  coincident M5-branes in a flat spacetime is given by an interacting  $(2, 0)$  superconformal theory in six dimensions [2]. The understanding of the dynamics of this system is of utmost importance. It will not only improve our understanding of the AdS/CFT correspondence for the  $AdS_7 \times S^4$  background [3]; in addition, as the problem involves a mathematical formulation of a self-duality equation for a non-abelian 3-form gauge field strength, one may

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suspect that it may have an impact on mathematical physics in a way similar to its lower dimensional cousin, the self-dual Yang–Mills equation [4].

On general grounds, the theory of multiple M5-branes does not have a free dimensionless parameter and is inherently non-perturbative. It does not mean that an action does not exist, though it does mean that the action will be of limited use, probably no more than giving the corresponding equation of motion. This is still very interesting since one can expect that non-trivial spacetime physics of M-theory could be learned from the physics of the solitonic objects of the worldvolume theory of M5-branes, much like the cases of M2-branes and D-branes. See for example, [5].

In a recent paper [1], a consistent self-duality equation of motion for a non-abelian tensor gauge field in six dimensions has been constructed and proposed to be the low energy equation of motion of the self-dual tensor field living on the worldvolume of a system of multiple M5-branes. The self-dual equation of motion proposed in [1] is meant to be an effective description for the M5-branes, just like the supergravity equation of motion provides an effective description for the M-theory. Recently it has been conjectured [6,7] that the 5d supersymmetric Yang–Mills theory can be used to provide a fundamental definition of the (2, 0) theory. This conjecture has been checked quite recently in [8] and it was shown that the 5d supersymmetric Yang–Mills theory is divergent at six loops and hence extra degrees of freedom is needed to provide a UV completion. Having a workable fundamental definition for the M5-branes system is highly desirable but very difficult, see also [9–11] for other proposals. We will be restricting ourselves with the effective description and hope something useful can be learned.

The non-abelian self-duality equation constructed in [1] generalizes the equation of motion for a single M5-brane of [12–15]. It was constructed in the gauge  $B_{5\mu} = 0$  ( $\mu = 0, \dots, 4$ ) and is a non-abelian generalization of the Henneaux–Teitelboim–Perry–Schwarz construction for the  $U(1)$  case [13,16]. The construction of [1] involves the introduction of a set of non-propagating non-abelian 1-form gauge fields which was motivated originally by the boundary analysis in [17] and further analyzed [18]. This aspect is very similar to the BLG [19] and ABJM model [20] of multiple M2-branes where a set of non-propagating Chern–Simons gauge fields was introduced in order to allow for a simple representation of the highly non-linear and non-local self-interactions of the matter fields of the theory.

The proposed self-duality equation reads

$$\tilde{H}_{\mu\nu} = \partial_5 B_{\mu\nu}, \quad (1)$$

where the gauge field  $A_\mu$  is constrained to be given by

$$F_{\mu\nu} = c \int dx_5 \tilde{H}_{\mu\nu}. \quad (2)$$

Here

$$H_{\mu\nu\rho} = D_{[\mu} B_{\nu\rho]} = \partial_{[\mu} B_{\nu\rho]} + [A_{[\mu}, B_{\nu\rho]}], \quad (3)$$

$$\tilde{H}_{\mu\nu} = \frac{1}{6} \epsilon_{\mu\nu\rho\sigma\tau} H^{\rho\sigma\tau}, \quad \epsilon_{01234} = -1, \quad (4)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \quad (5)$$

All fields are in the adjoint representation of the Lie algebra of the gauge group  $G$ , and  $c$  is a free parameter. Our convention for the Lie algebra are:  $[T^a, T^b] = i f^{abc} T^c$ ,  $F_{\mu\nu} = i F_{\mu\nu}^a T^a$ ,  $A_\mu = i A_\mu^a T^a$  and  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c$ .

Evidence that this self-duality equation describes the physics of multiple M5-branes was provided in [1], and further in [21–23]. In [21,22], non-abelian self-dual string solutions were constructed and a precise agreement [22] of the field theory results and the supergravity descriptions [24] was found. Moreover, it was found that the constant  $c$  is fixed by quantization condition of the self-dual strings solution of the theory. This is satisfying as otherwise  $c$  would be a free dimensionless constant in the theory and hence contradicts with what we know about M5-branes in flat space. In [23], non-abelian wave configurations which are supported by Yang–Mills instanton were constructed and they were found to match up nicely with the description of M-wave on the worldvolume of M5-branes system.

One thing interesting about the self-dual string solutions constructed in [21,22] is that the auxiliary gauge field is always given by a magnetic monopole which gives rise to the charge of the self-dual string. This was shown to be case for the original Perry–Schwarz self-dual string and the Wu–Yang self-dual string [21], as well as for the generalized Wu–Yang self-dual string [22], with the corresponding monopole configurations given by the Dirac monopole, the Wu–Yang monopole and the generalized Wu–Yang monopole. It is natural to ask if this connection with monopole is a general feature of the non-abelian self-dual string. In this paper, we show that this is indeed the case. This result is potentially interesting as, given this rather explicit connection between BPS monopole and self-dual string, one may be able to provide a Nahm like construction for non-abelian self-dual string, which has been speculated and analyzed by other authors [25].

In the next section, we provide a general formalism for the construction of non-abelian self-dual string starting from an 1/2-BPS monopole solution in four dimensions. In Section 3, we show that one can recover the previously constructed self-dual string solutions with this new formalism. We also construct, for the 't Hooft–Polyakov monopole [26,27], a singular solution that describes two finitely separated M5-branes meeting midway in between. The paper is concluded with some further discussions in Section 4.

## 2. A general construction of self-dual strings in terms of BPS monopoles

In this section, we give a general construction for self-dual strings solutions to the non-abelian self-duality equations (1), (2). We will be interested in static configurations with the self-dual string being infinite long straight line, say, in the  $x^4$ -direction. As a result, physical properties of the system are independent of  $x^0$  and  $x^4$ .

Let us consider an ansatz with the following non-vanishing components of the  $B$ -field:

$$B_{ij}, \quad B_{04} := -\phi. \quad (6)$$

The non-vanishing components of  $H$  are

$$H_{ijk} = D_{[i} B_{jk]}, \quad H_{5ij} = \partial_5 B_{ij}, \quad (7)$$

$$H_{04i} = -D_i \phi, \quad H_{045} = -\partial_5 \phi, \quad (8)$$

and the self-duality equation (1) reads

$$\partial_5 \phi = -\frac{1}{2} \epsilon_{ijk} D_i B_{jk}, \quad (9)$$

$$D_k \phi = \frac{1}{2} \epsilon_{kij} \partial_5 B_{ij}. \quad (10)$$

We remark that for  $U(1)$  gauge group, the self-duality equations (9), (10) are precisely the same as the BPS equations of Howe–Lambert–West [28] for the (2, 0) M5-branes theory in

the case when only a single worldvolume scalar field  $\phi := \phi^6$  is turned on. In [21,22], it was postulated that (7), (8) are the BPS equation for the non-abelian (2, 0) theory.

### 2.1. General construction

Eq. (10) can be integrated and solved by

$$B_{ij} = \epsilon_{ijk} D_k \Phi, \quad (11)$$

where  $\Phi$  is related to  $\phi$  by

$$\phi = \partial_5 \Phi. \quad (12)$$

Substitute (11) into (9), we obtain immediately the 4-dimensional covariant Laplace equation

$$(\partial_5^2 + D_i^2) \Phi = 0. \quad (13)$$

The constraint (2) now reads

$$F_{ij} = c \epsilon_{ijk} D_k \tilde{\Phi}, \quad (14)$$

$$F_{04} = -c(\phi(x^5 = \infty) - \phi(x^5 = -\infty)), \quad (15)$$

where

$$\tilde{\Phi}(x^i) := \Phi(x^i, \infty) - \Phi(x^i, -\infty). \quad (16)$$

Eq. (14) can be solved immediately by noticing that it takes precisely the form of the BPS equation of a magnetic monopole, with  $c\tilde{\Phi}$  being the adjoint Higgs scalar field of the Yang–Mills theory. We note in passing that  $\tilde{\Phi}$  satisfies the 3-dimensional covariant Laplace equation

$$D_i^2 \tilde{\Phi} = 0. \quad (17)$$

For excellent reviews of monopole, see for example, [29–31]. As for solving (14), we can distinguish two cases. For a regular solution  $\Phi(x^i, x^5)$  with well defined limits  $\Phi(x^i, \pm\infty)$ , we have

$$\phi(x^5 = \infty) = \phi(x^5 = -\infty) = 0. \quad (18)$$

In this case the constraint (15) can be solved conveniently by having  $A_0 = A_4 = 0$ . We will also be interested in solutions with singularities such that non-vanishing values of  $\phi(x^5 = \pm\infty)$  are allowed. In this case, the constraint (15) is solved by  $A_4 = 0$ ,  $A_0 = c(\phi(x^5 = \infty) - \phi(x^5 = -\infty))x^4$ .

Base on the above observation, a general self-dual string solution of the self-duality equations (9) and (10) can be constructed entirely in terms of the solution of the covariant Laplace equation (13) for an adjoint scalar field. Our algorithm is to start with *any* BPS monopole configuration  $(A_i, \Phi^{(0)})$  of the Yang–Mills Higgs system as a seed, and looks for a solution of the covariant Laplace equation (13) with the boundary condition

$$c\tilde{\Phi}(x^i) = \Phi^{(0)}(x^i), \quad (19)$$

where  $\tilde{\Phi}(x^i)$  is given by (16). Then a self-dual string solution solving the self-duality equations (9), (10) and the constraints (14), (15) is given by

$$B_{ij} = \epsilon_{ijk} D_k \Phi, \quad (20)$$

$$\phi = \partial_5 \Phi. \quad (21)$$

In addition, we also need to specify certain boundary conditions on  $\phi$  (see (34) below) so that the field strengths decrease fast enough at infinity in order for the charge to be well defined. This will be examined next.

## 2.2. Charges

A general feature of the non-abelian monopole is that the gauge symmetry  $G$  is broken down asymptotically to a little group  $H$  by the large  $r$  values of the scalar field  $\tilde{\Phi}$ .  $H$  generally contains a  $U(1)$  factor generated by

$$T := T^a \hat{n}^a, \quad (22)$$

where the unit vector  $\hat{n}^a$  is determined by the asymptotic configuration of  $\tilde{\Phi}$ . As a result, our non-abelian self-dual string solution inherits at large  $r$  (and for any  $x^5$ ) the same unbroken gauge symmetry. This allows us to define an asymptotic  $U(1)$  field by a projection

$$B_{\mu\nu} := B_{\mu\nu}^a \hat{n}^a. \quad (23)$$

It has the asymptotic  $U(1)$  field strength

$$H_{\mu\nu\lambda} := \partial_{[\mu} B_{\nu\lambda]} = D_{[\mu} B_{\nu\lambda]}^a \hat{n}^a. \quad (24)$$

The magnetic and electric charges (per unit length) of our self-dual string solution is then given by

$$P = Q = \frac{1}{2\pi^2} \int_S H, \quad (25)$$

where  $S$  is the boundary surface of a large volume  $V$  in  $\mathbf{R}^4$  (of  $x^1, x^2, x^3, x^5$ ) containing the self-dual string configuration. We claim that the self-dual string charge is indeed given by charge of the seed BPS monopole

$$P = Q = \frac{c}{2\pi^2} \int_{S^2} F, \quad (26)$$

where  $S^2$  is a 2-sphere in the space of  $x^1, x^2, x^3$  containing the BPS monopole and  $F := F^a \hat{n}^a$  is the asymptotic  $U(1)$  field strength for the non-abelian monopole.

To see this, let us take  $V$  to be the hypercube with boundary defined by the eight  $\mathbf{R}^3$ -hyperplanes:

$$S: \quad x^5 = \pm\infty, \quad \text{or} \quad x^i = \pm\infty. \quad (27)$$

This gives

$$2\pi^2 P = 2\pi^2 Q = I_5 + I_1 + I_2 + I_3, \quad (28)$$

where

$$I_5 := \int_{\mathbf{R}^3} H_{123} \Big|_{x^5=-\infty}^{x^5=\infty} dx^1 dx^2 dx^3, \quad (29)$$

and

$$I_1 := \int_{\mathbf{R}^3} H_{523} \Big|_{x^1=-\infty}^{x^1=\infty} dx^5 dx^2 dx^3, \quad (30)$$

with  $I_2$  and  $I_3$  defined similarly. To evaluate the integrals, we compute the field strengths. We have

$$\begin{aligned} H_{123} \Big|_{x^5=-\infty}^{x^5=\infty} &= [\partial_1 (B_{23}^a(x^5 = \infty) - B_{23}^a(x^5 = -\infty)) + (123 \text{ cyclic})] \hat{n}^a \\ &= c [\partial_1 F_{23}^a + (123 \text{ cyclic})] \hat{n}^a \\ &= c (\partial_1 F_{23} + (123 \text{ cyclic})), \end{aligned} \quad (31)$$

where we have used (2) and  $F := F^a \hat{n}^a$  is the asymptotic  $U(1)$  field strength for the non-abelian monopole. Therefore (31) is non-vanishing at the position of the monopoles and we have

$$I_5 = \frac{c}{2\pi^2} \int_{S^2} F. \quad (32)$$

As for the other integrals  $I_1, I_2, I_3$ , we require that

$$H_{523} = D_1 \phi^a \hat{n}^a \rightarrow 0 \quad \text{as } x^1 \rightarrow \pm\infty \quad (33)$$

tends to zero fast enough so that the integral  $I_1$  vanishes. Similar considerations apply for  $I_2$  and  $I_3$ . In total, we require that

$$D_i \phi^a \rightarrow 0 \quad \text{as } x^i \rightarrow \pm\infty. \quad (34)$$

As a result, we obtain (26) as claimed.

### 3. Examples

In this section, we give explicit examples to illustrate our general construction. We first show that the known solutions such as the Perry–Schwarz self-dual string [13] for the  $U(1)$  theory and the Wu–Yang monopole string [21,22] for the non-abelian theory can both be considered as examples of our general construction. We then consider the 't Hooft–Polyakov BPS monopole and show that no acceptable self-dual string solution can be constructed. And we discuss the possible physical reason behind this negative result.

#### 3.1. Perry–Schwarz self-dual string and Wu–Yang monopole self-dual string

The Perry–Schwarz self-dual string solution is given by the following configuration of  $B$ -field:

$$B_{ij}^{(\text{PS})} = -\frac{\beta}{2} \frac{\epsilon_{ijk} x^k}{r^3} \left[ \frac{x^5 r}{\rho^2} + \tan^{-1} \left( \frac{x^5}{r} \right) \right], \quad (35)$$

$$B_{04}^{(\text{PS})} = -\frac{\beta}{2\rho^2}, \quad (36)$$

where  $r^2 = (x^i)^2$  and  $\rho^2 = (x^i)^2 + (x^5)^2$ . This is not the original form Perry–Schwarz obtained, but, as was shown in [21], is gauge equivalent to it. In this abelian case,  $A_i$  decoupled from the Laplace equation (13) and so we have to solve

$$(\partial_5^2 + \partial_i^2)\Phi = 0. \quad (37)$$

The 4-dimensional Laplace equation has the solution  $1/\rho^2$ . But this has trivial boundary condition. It is easy to see that it's integral

$$\Phi := \int_{-\infty}^{x^5} \frac{\beta}{2\rho^2} dy^5 = \frac{\beta}{2r} \left[ \tan^{-1} \left( \frac{x^5}{r} \right) + \frac{\pi}{2} \right] \quad (38)$$

also satisfies the Laplace equation (13). Moreover, it satisfies the boundary condition (19) with

$$\tilde{\Phi} = \frac{\beta\pi}{2r}. \quad (39)$$

This gives

$$b_i := \frac{1}{2} \epsilon_{ijk} F_{jk} = \partial_i (c\tilde{\Phi}) = -\frac{c\beta\pi}{2} \frac{x^i}{r^3}. \quad (40)$$

This is precisely the magnetic field strength for a Dirac monopole of charge  $n$  if

$$\beta = -\frac{2n}{c\pi}. \quad (41)$$

Our (38) reproduces precisely the Perry–Schwarz  $B_{ij}$ . By construction

$$\phi = \frac{\beta}{2\rho^2} + v. \quad (42)$$

In the paper [21], a self-dual string solution for the  $SU(2)$  theory is obtained with its auxiliary Yang–Mills field given by the generalized Wu–Yang monopole configuration:

$$A_i^a = -\epsilon_{aik} \frac{x^k}{r^2}. \quad (43)$$

It is easy to see that for the ansatz,

$$\Phi = \frac{ix^a T^a}{r} \varphi, \quad (44)$$

where  $T^a$ ,  $a = 1, 2, 3$ , are the  $SU(2)$  generators,

$$[T^a, T^b] = i\epsilon^{abc} T^c. \quad (45)$$

The covariant Laplace equation simplifies to

$$(\partial_5^2 + \partial_i^2)\varphi = 0. \quad (46)$$

We can take  $\varphi$  to be given by (38). This gives

$$\tilde{\Phi} = \frac{ix^a T^a}{r} \tilde{\varphi}, \quad \text{where } \tilde{\varphi} := \frac{\beta\pi}{2r}, \quad (47)$$

$$F_{ij} = \epsilon_{ijk} D_k (c\tilde{\Phi}) = -\frac{c\beta\pi}{2} \frac{\epsilon_{ijk} x^k}{r^3} \frac{ix^a T^a}{r} \quad (48)$$

and

$$\begin{aligned}
 B_{ij} &= \epsilon_{ijk} D_k \Phi = B_{ij}^{(\text{PS})} \frac{ix^a T^a}{r}, \\
 B_{04} &= -\partial_5 \Phi = B_{04}^{(\text{PS})} \frac{ix^a T^a}{r}.
 \end{aligned} \tag{49}$$

By construction

$$\phi = \frac{ix^a T^a}{r} \left( \frac{\beta}{2\rho^2} + v \right). \tag{50}$$

Similarly, one can reproduce the generalized Wu–Yang monopole string obtained in [22]. Note that, up to a gauge transformation, the scalar  $\phi$  as well as the field strength are  $SO(4)$  rotational invariant.

### 3.2. 't Hooft–Polyakov monopole self-dual strings

Next we consider the 't Hooft–Polyakov monopole. The 't Hooft–Polyakov monopole solution is a topological soliton of the non-abelian  $SU(2)$  gauge Yang–Mills gauge theory which is non-singular and carry a magnetic charge since it has an asymptotic behavior similar to that of a Dirac monopole. In the BPS limit, the 't Hooft–Polyakov monopole satisfies the first-order BPS equation

$$F_{ij} = \epsilon_{ijk} D_k \Phi^{(\text{HP})} \tag{51}$$

and has an analytic solution given in terms of elementary functions  $k(r)$ ,  $h(r)$  as

$$A_0^a = 0, \quad A_i^a = -\epsilon_{aik} \frac{x^k}{r^2} (1 - k(r)), \tag{52}$$

$$\Phi^{(\text{HP})a} = \frac{x^a}{r^2} h(r), \tag{53}$$

where

$$k(r) = \frac{r}{\sinh r}, \quad h(r) = r \coth r - 1. \tag{54}$$

At large  $r$ , it is

$$k \rightarrow 0, \quad h \rightarrow r, \quad \text{for } r \rightarrow \infty. \tag{55}$$

This means the scalar field approaches asymptotically a constant vacuum expectation value with the magnitude ( $|B| := \sqrt{B^a B^a}$ )

$$|\Phi^{(\text{HP})}| \rightarrow 1 \tag{56}$$

and an unbroken  $U(1)$  gauge symmetry can be identified there by projection. We have taken, for simplicity, the vacuum expectation value of the scalar field to be equal to 1. The unbroken  $U(1)$  is generated by the generator

$$T = \frac{ix^a T^a}{r} \tag{57}$$

and the asymptotic  $U(1)$  field strength is precisely equal to that of the Dirac monopole of unit charge

$$F := F^a \frac{ix^a}{r} = F^{(\text{Dirac})} \tag{58}$$



and so the 't Hooft–Polyakov monopole carries a unit monopole charge

$$P = \frac{1}{2\pi} \int_{S^2} F = 1. \quad (59)$$

Despite the existence of a non-zero charge, the solution is regular everywhere. In fact, for small  $r$ ,

$$k \rightarrow 1, \quad h \rightarrow -1 \quad (60)$$

and so

$$A_\mu^a \rightarrow 0, \quad |\Phi^{(\text{HP})}| \rightarrow 0. \quad (61)$$

Now, let us start with the BPS monopole configuration (52) and apply the general construction in the last section. As the 't Hooft–Polyakov monopole is radial symmetric, the resulting self-dual string should be at least  $SO(3)$  invariant. This implies that  $\Phi$  should be of the form

$$\Phi = \frac{ix^a T^a}{r} \varphi(x^5, r). \quad (62)$$

The covariant Laplace equation (13) then reduce to the single partial differential equation on  $\varphi$

$$\left( \partial_i^2 + \partial_5^2 - \frac{2k^2(r)}{r^2} \right) \varphi = 0. \quad (63)$$

Substituting

$$\varphi = \frac{H(r, x^5)}{r} \quad (64)$$

and Eq. (63) is further reduced to

$$H'' + \ddot{H} - \frac{2}{\sinh^2 r} H = 0, \quad (65)$$

where  $H' = \partial_r H$  and  $\dot{H} = \partial_5 H$ . The differential equation is solved generally by

$$H = \coth r \times f - \frac{\partial f}{\partial r}, \quad (66)$$

provided  $f = f(x^5, r)$  satisfies the 2-dimensional Laplace equation

$$(\partial_r^2 + \partial_5^2) f = 0. \quad (67)$$

General solution to the Laplace equation (67) can be readily written down using the theory of complex variables. The less trivial part is to find solution so that the boundary conditions (19) and (34) are satisfied. To satisfy (19), it is needed that

$$c(f(x^5 = \infty) - f(x^5 = -\infty)) = r. \quad (68)$$

Let us first consider solution that is finite at infinite  $x^5$ , corresponds to the situation of (18). Such regular solution can be represented by the Fourier series

$$cf = \int_0^1 dp \alpha(p) \frac{\sin(px^5)}{p} \frac{\sinh(pr)}{p}, \quad (69)$$

without including the linear term in  $x^5$ . It is easy to see that the condition (68) is satisfied with  $\alpha(0) = c/\pi$  since  $\frac{1}{p} \sin(px^5) \rightarrow \pm \frac{\pi}{2} \delta(p)$  as  $x^5 \rightarrow \pm\infty$ . This solution however blows up ( $\sim e^r$ ) at large  $r$  and is nonphysical for a self-dual string. For our problem at hand, the most general solution to (67) is of the form

$$cf = \int_0^1 dp \left( \alpha(p) \frac{\sin(px^5)}{p} e^{-pr} + \beta(p) \cos(px^5) e^{-pr} \right), \quad (70)$$

where we have not included terms  $e^{pr}$  for  $p > 0$ , and without loss of generality, we have taken the upper limit of integration to be 1. Noticing that  $\cos(px^5) \rightarrow 0$  as  $x^5 \rightarrow \pm\infty$ , we see that it is impossible to satisfy (19) for any choice of the coefficients  $\alpha(p)$  and  $\beta(p)$ .

Let us try to understand the physical reasons for the absence of self-dual string solution in this case. According to our general analysis in Section 2.2, the charge of the self-dual string is given by the charge of the BPS monopole, which for the case of 't Hooft–Polyakov monopole is one. As we mentioned above, it is quite suggestive that the self-duality equation (9), (10) coincides with the BPS equation of the (2, 0) theory when only a single scalar field  $\phi$  is turned on. If this conjecture is indeed correct, then the self-dual string solution we were seeking for should be  $SO(4)$  rotational invariant as it should be a minimal energy configuration (being supersymmetric) and has a single center (having unit charge). However an  $SO(4)$  invariant solution corresponding to a unit charge self-dual string has already been constructed previously for general  $SU(N_5)$  gauge group [22]. This solution was based on the Wu–Yang monopole and the scalar profile of this solution has been found to match precisely with the radius-transverse distance relation obtained from a supergravity analysis of the M2-brane spike intersecting a system of M5-branes [24]. It would be puzzling if one is able to find a different field theory solution that describes an M2-brane ending on a system of M5-brane.

Physically, there are other interesting configurations one may have for a system of M2-branes and M5-branes. For the analogous system of D-branes, one interesting configuration is to have a fundamental string or D-string stretching between two D3-branes as described by the wormhole like solution constructed by Callan–Maldacena [32] and Gibbons [33] (see Fig. 1(a)). This configuration is, however, non-BPS as the two D3-branes are smoothly connected with each other through the throat and hence are oppositely orientated. Another interesting configuration considered by Hashimoto [34] is for the two D3-branes to meet at precisely the point where the throat size  $r$  shrinks to zero (see Fig. 1(b)). The configuration is BPS as the two D3-branes are connected through a singularity at  $r = 0$  and is described by the 't Hooft–Polyakov BPS monopole in the  $SU(2)$  SYM theory.

Back to our problem of finding solution of Eqs. (9), (10). In addition to the spike solution [21,22] which describes a bunch of M2-branes ending on a system of M5-branes, we wish to find a configuration similar to the one of Hashimoto for D-branes.<sup>1</sup> To get this solution, we will have to relax ourselves to consider solution with singularity. So far we have considered solution that is finite at infinite  $x^5$ . In order to describe the presence of two M5-branes, in view of (18), one has to give up the requirement of  $\Phi$  being finite at infinite  $x^5$ . Therefore let us relax the regularity condition of  $\Phi$  at infinite  $x^5$  and consider a solution of the form

<sup>1</sup> Obviously we do not expect to find a solution to our Eqs. (9), (10) to describe an M2-brane suspended between two M5-branes as this configuration is non-BPS.

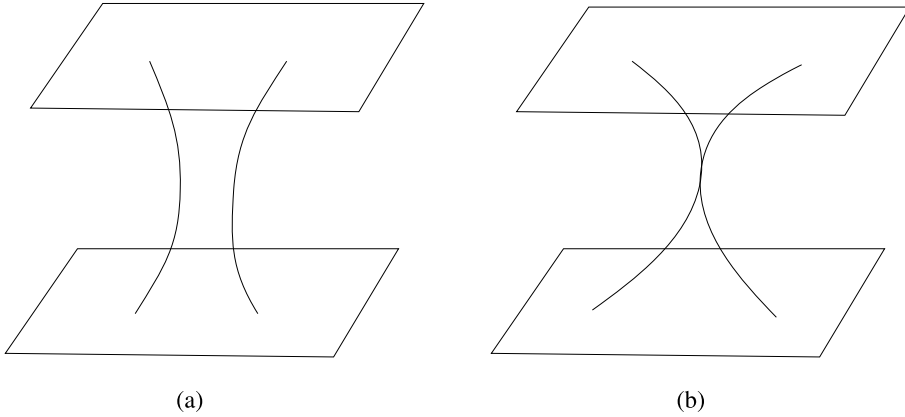


Fig. 1. A string stretches between a pair of parallel D3-branes. (a) is described by a wormhole like solution. (b) is singular at  $r = 0$  and is described by a BPS monopole.

$$f := \begin{cases} x^5 r + r/c, & x^5 \geq 0, \\ -x^5 r, & x^5 < 0. \end{cases} \quad (71)$$

We could also add to it a Fourier series part but this would not affect the discussion below. It is clear that the corresponding  $\Phi$  is discontinuous at  $x^5 = 0$ . Also it is singular at infinite  $x^5$ . To regular the singularity, we can put the system in a box of size  $L$ :  $-L \leq x^5 \leq L$ . Then the boundary condition (68) is satisfied. One can check the boundary condition (34) is also satisfied. This solution looks rather trivial. What does it correspond to physically? To see this, note that

$$\phi = \Phi^{(\text{HP})}(r) \times \begin{cases} 1, & x^5 \geq 0, \\ -1, & x^5 < 0. \end{cases} \quad (72)$$

We could diagonalize this field so that its eigenvalues can be interpreted as the positions of the M5-branes. As is well known from the analysis of the 't Hooft–Polyakov monopole, this can be achieved locally with an appropriate gauge transformation and we find

$$\phi = \pm \left( \coth r - \frac{1}{r} \right) \times \begin{cases} 1, & x^5 \geq 0, \\ -1, & x^5 < 0. \end{cases} \quad (73)$$

This solution is illustrated in Fig. 2. Asymptotically as  $r \rightarrow \infty$ ,  $\phi$  approaches the constant values  $\pm 1$ , corresponds to having two M5-branes placing at the constant positions  $\phi = \pm 1$ . Near  $r = 0$ ,  $|\phi| \sim r/3$ , meaning the cross-section of the M5-brane shrink linearly to zero as one approaches the point halfway between the two M5-branes. This solution describes a pair of parallel M5-branes with their respective M2-brane spikes meeting at halfway between the two M5-branes.

#### 4. Discussions

In this paper, we have given a general formalism for the construction of a class of non-abelian self-dual string solution whose auxiliary gauge field is given by a BPS magnetic monopole. The charge of the self-dual string is given, up to a proportional constant, by the charge of the monopole. We have shown that previously found  $SO(4)$  invariant solution can be recovered by

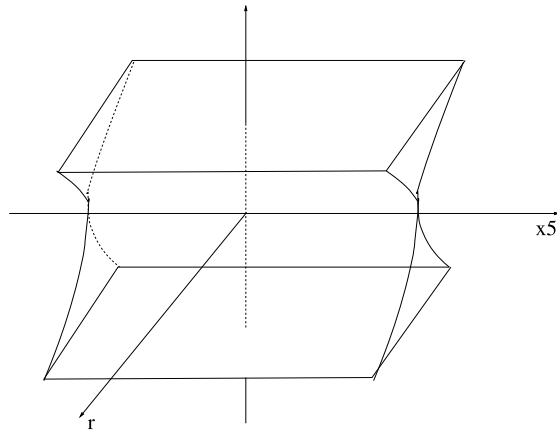


Fig. 2. Two parallel M5-branes meeting midway in between.

this formalism. We have also constructed, based on the 't Hooft–Polyakov monopole, a singular solution that describes two finitely separated M5-branes meeting midway in between.

We remark that a class of string solitons of the models of [35] has been found in the papers [36,37]. These solutions are smooth and have charges supported by instanton configurations. This is to be compared with the behavior of the string solution obtained in [13,28], and those obtained in [21,22] and here, which are singular at the position of the string. A better understanding of these solutions and their differences will help us to understand better the differences of these models.

Having analyzed the charge one sector, it is natural to consider the higher charges case. Rather explicit expressions exist for the charge two monopoles [38] and the corresponding self-dual string solution is expected to have two centers and hence  $SO(3)$  invariant. It would be interesting to employ the present formalism to construct this self-dual string solution.

For higher charges, the BPS monopole configurations are given systematically by the Nahm construction

$$\tilde{\Phi}^{mn}(x^i) = \int_{-v/2}^{v/2} ds w_m^\dagger(s, x^i) s w_n(s, x^i), \quad (74)$$

$$A_i^{mn}(x^i) = -i \int_{-v/2}^{v/2} ds w_m^\dagger(s, x^i) \partial_i w_n(s, x^i), \quad (75)$$

where  $w_m(s, x^i)$  are normalizable solutions to a certain linear differential equation in  $s$  with coefficients constructed from the Nahm data and  $x^i$ . The relation between a class of non-abelian self-dual strings and BPS monopoles in four dimensions revealed in this paper suggests that one may be able to generalize the Nahm construction for monopoles to the non-abelian self-dual strings. It is tempting to interpret the variable  $s$  as the dimension  $x^5$ . On the other hand, the inclusion of a constant NSNS  $B$ -field in the BPS monopole simple amounts to a constant shift in the Nahm equation. The availability of the generalized Nahm construction may allow one to include a constant  $C$ -field in a simple manner and check against the quantum geometry [39,40] obtained from other pictures of the system. We leave this interesting issue for future discussions.

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